Foreign Reserve Management

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Motivation

Over the past 20 years massive increase in foreign reserves holdings by Central Banks around the world.
Motivation (ctd)

Why do central banks hold foreign reserves?

1. *Precautionary motive*: reserves used as a buffer for bad shocks

2. *Exchange rate management*: reserves used to achieve a policy for nominal exchange rates
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How should central banks manage their portfolio?

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This paper: Given exchange rates and monetary policy objectives, How should a Central Bank manage its reserve portfolio?
CB has a monetary policy objective: \( \{i, e_t, e_{t+1}\} \)

Suppose that \((1 + i) \frac{e_t}{e_{t+1}} > (1 + i^*)\) (needs limited arbitrage)
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- Euler equation in the domestic market

\[
u'(c_t) = \beta \left[ (1 + i)\frac{e_t}{e_{t+1}} \right] u'(c_{t+1})
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- **Euler equation in the domestic market**

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- **Unique** consumption profile \( \Rightarrow \) Requires foreign reserve accumulation (but no portfolio choice) by the CB
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Policy has two costs

- Current consumption is too low
- Resource loss, as foreigners exploit interest differential
Foreign reserve management with uncertainty

With uncertainty, consider similar policy violating interest parity
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- **Multiple** consumption profiles consistent with same targets
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- **Multiple** consumption profiles consistent with same targets

- CB can implement *any* of them by managing its foreign reserves portfolio
  - Tilts consumption towards the future, as before
  - But can also *change consumption across states*
• Thus CB has more options with uncertainty

For example:

• A negative covariance between the appreciation and future marginal utility boosts $c_t$ for same targets:

$$u'(c_t) = \beta \mathbb{E} \left[ (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right]$$
With uncertainty (continued)

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- But other domestic asset prices are affected

  $\Rightarrow$ Potentially larger resource loss: *foreigners exploit the best return differential*
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Trade-off: consumption smoothing vs resource losses
Resolving the trade-off

- When potential capital inflows are small – resource losses are small
  - Optimal to focus on consumption smoothing
  - Reserve management goal: increase consumption in states where currency appreciates
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- Optimal to focus on consumption smoothing
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When potential capital inflows are large – resources losses are large

- Optimal to focus on minimizing resource losses
- Purchase relatively safe foreign portfolio
Framework

- Two-period model, $t \in \{1, 2\}$
  - Small open economy (central bank + households)
  - International Financial Market
  - Foreign Intermediaries

- Uncertainty realized at $t = 2$
  - $s \in S \equiv \{s_2, \ldots, s_N\}, \pi(s)$

- One (tradable) good, law of one price, foreign price normalized to 1
Asset markets: complete but segmented

International financial markets (IFM)
- Full set of Arrow-Debreu securities in foreign currency:
  - Security $s$: 1 unit of foreign currency in state $s$, 0 otherwise
  - Price $q(s)$ in terms of foreign currency at $t = 1$

Domestic financial market
- Full set of Arrow-Debreu securities in domestic currency
  - Security $s$: 1 unit of domestic currency in state $s$, 0 otherwise
  - Price $p(s)$ in terms of domestic currency at $t = 1$

Foreign Intermediaries
- Trade securities with SOE & IFM and have limited capital
Households

- Endowment: \((y_1, \{y_2(s)\})\), transfers: \(\{T_2(s)\}\)

\[
\max_{c_1, \{c_2(s), a(s), f(s)\}} \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

subject to:

\[
y_1 = c_1 + \sum_{s \in S} \left[ q(s)f(s) + p(s)\frac{a(s)}{e_1} \right]
\]

\[
y_2(s) + T_2(s) + f(s) + \frac{a(s)}{e_2(s)} = c_2(s) \quad \forall s \in S
\]

\[
f(s) \geq 0, \quad \forall s \in S
\]

\(e_1, e_2(s)\): exchange rates at \(t = 1\) and \(t = 2\)

\(f(s), a(s)\): holdings of foreign and domestic security \(s\)
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\(e_1, e_2(s)\): exchange rates at \(t = 1\) and \(t = 2\)

\(f(s), a(s)\): holdings of foreign and domestic security \(s\)
Foreign Intermediaries

- Endowed with capital $\bar{w}$

\[
\max_{\{d_1^*, d_2^*(s), a^*(s), f^*(s)\}} \quad d_1^* + \sum_{s\in S} \pi(s) \Lambda(s) d_2^*(s)
\]

subject to:

\[
\bar{w} = d_1^* + \sum_{s\in S} p(s) \frac{a^*(s)}{e_1} + \sum_{s\in S} q(s) f^*(s)
\]

\[
d_2^*(s) = \frac{a^*(s)}{e_2(s)} + f^*(s) \quad \forall s \in S
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\[
f^*(s) \geq 0 \quad a^*(s) \geq 0, \quad \forall s \in S
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Consider $\Lambda(s) = \frac{q(s)}{\pi(s)}$ (same SDF as IFM)
Foreign Intermediaries

- Endowed with capital $\tilde{w}$

$$\max_{\{d_1^*, d_2^*(s), a^*(s), f^*(s)\}} \quad d_1^* + \sum_{s \in S} \pi(s) \Lambda(s) d_2^*(s)$$

subject to:

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$$d_2^*(s) = \frac{a^*(s)}{e_2(s)} + f^*(s) \quad \forall s \in S$$

$$f^*(s) \geq 0 \quad a^*(s) \geq 0, \quad \forall s \in S$$

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Central Bank

- CB has an objective for the nominal interest rate and exchange rates that we take as given: \((i, e_1, \{e_2(s)\})\)

\[
1 + i = \left( \sum_{s \in S} p(s) \right)^{-1} \quad \text{(NIRC)}
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CB achieves its objective by managing its balance sheet: invest \(\{A(s), F(s)\}\); and transfers \(\{T_2(s)\}\) to households, subject to budget constraints.
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Same portfolio of securities as households (no hedging motive)
Arbitrage return for security $s$:

\[ \kappa(s) \equiv \frac{e_1}{e_2(s)p(s)} - 1 \]

$\kappa(s) > 0 \Rightarrow$ domestic security paying in state $s$ yields higher return

- Households: borrow up to limit in foreign currency security and invest in domestic one.
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- Intermediaries: invest all available funds in security that delivers highest return.
• \textbf{Arbitrage return} for security $s$:

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• Households: borrow up to limit in foreign currency security and invest in domestic one.

• Intermediaries: invest all available funds in security that delivers highest return. Let $\bar{\kappa} \equiv \max_s \{k(s)\}$

$\Rightarrow$ Profits $\bar{\kappa} \times \bar{w}$
Characterizing equilibria: Resource constraint

Profits for intermediaries are losses for the SOE

\[(y_1 - c_1) + \sum_{s \in S} q(s)[y_2(s) - c_2(s)] = \bar{\kappa}\bar{\omega}\]
CB objective \((i, e_1, \{e_2(s)\})\) determines the risk-adjusted return differential between the risk-free domestic bond and the foreign one

\[
\Delta(i) \equiv \mathbb{E} \left[ \Lambda(s) \left( \frac{e_1}{e_2(s)} (1 + i) - (1 + i^*) \right) \right]
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Central bank objective and interest parity

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If \(\Delta(i) > 0\), domestic assets \textit{dominate} foreign assets. The opposite happens when \(\Delta(i) < 0\).
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Focus on regime in which \(\Delta(i) > 0\)

- More likely if currency expected to appreciate or safe heaven.
- Requires some securities to have \(\kappa(s) \geq \Delta(i)\)
$\Delta(i) > 0$ implies that capital flows in & $c_1$ is low
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From BOP equation, CB needs to buy *some* foreign assets

\[
c_1 - y_1 = \sum_s \frac{p(s)}{e_1} a^*(s) - \sum_s q(s) [f(s) + F(s)]
\]
On the Need of Central Bank Intervention

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Households are privately *unwilling* (but able) to make these trades and *unable* to undo them.
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Which assets $\{F(s)\}$ should CB buy?
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Which assets $\{F(s)\}$ should CB buy?

- Potential size of capital flows is key
- Today: two cases
Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^* \pi(s)$

- Higher $\kappa(s)$ in states in which exchange rate appreciates
Financially closed economy

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- Nominal bond is too attractive $\Rightarrow$ “excessive” savings

- Key idea: promise low marginal utility (i.e., high $c_2, \kappa$) when nominal bond pays more (i.e., $e_2$ appreciates).
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NIRC binds from below:

$$\frac{1 + i^*}{1 + i} \geq \mathbb{E} \left( \frac{e_1}{e_2(s)} \right) \mathbb{E} \left( \frac{1}{1 + \kappa(s)} \right) + \text{Cov} \left( \frac{e_1}{e_2(s)}, \frac{1}{1 + \kappa(s)} \right)$$

- To reduce average intertemporal distortion $\sim \mathbb{E}[\kappa(s)]$, increase intratemporal distortions.
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\]

- To reduce average *intertemporal* distortion \( \sim \mathbb{E}[\kappa(s)] \), increase *intratemporal* distortions.
From arbitrage gaps to reserves, $\kappa(s) \rightarrow F(s)$

Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

\[ F(s) = c(s) - y(s) \]
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If the exchange rate appreciates in good times and assuming that output volatility is low:

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Financially open economy (large $\bar{w}$)

Recall losses: $\max_s \{ \kappa(s) \} \bar{w}$
Recall losses: $\max_s \{\kappa(s)\} \bar{\omega}$

$$\min_{\{\kappa(s)\}_{s \in S}} \left\{ \max_s \{\kappa(s)\} \right\}$$

$$s.t. \quad 0 \leq 1 + i - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)}$$

(NIRC)
Financially open economy (large $\bar{w}$)

Recall losses: $\max_s \{ \kappa(s) \} \bar{w}$

$$\min \{ \kappa(s) \}_{s \in S} \left\{ \max_s \{ \kappa(s) \} \right\}$$

s.t. $0 \leq 1 + i - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)}$ (NIRC)

- Optimal policy calls for equal gaps $\kappa(s) = \kappa \forall s$
  - only allocation in which intermediaries demand risk-free bonds
- Some leeway about CB portfolio, as long as it is relatively safe
Conclusion

- Developed a framework to analyze the reserve management problem for a CB with nominal objectives
- Uncover trade-off for reserve management, based on a risk-channel
- Show that foreign reserve management can play an important and independent role when traditional monetary policy tools are constrained or devoted to alternative objectives

Agenda

- Implementation with specific assets (e.g. bonds and equity)
- Capital controls on outflows
- Closed economy implications
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

s.t.
\[
y_1 - c_1 - \sum_s q(s) c_2(s) = L^* (\{\kappa(s)\}, \bar{\kappa})
\]

\[
1 - \sum_s \frac{q(s) e_1}{(1 + \kappa(s)) e_2(s)} = i
\]

\[
1 + \kappa(s) = \frac{q(s) u'_1(c_1)}{\beta \pi(s) u'(c_2(s))} \quad \forall s
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y_1 - c_1 - \sum_s q(s) c_2(s) = L^\ast(\{\kappa(s)\}, \bar{\kappa}) \quad \text{(IRC)}
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1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \quad \text{(\( \kappa(s) \))}
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The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V(\tilde{\kappa}) = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s)u(c_2(s)) \right\}
\]

s.t. \( y_1 - c_1 - \sum q(s)c_2(s) = L^*(\{\kappa(s)\}, \tilde{\kappa}) \) \hspace{1cm} (IRC)

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \quad \hspace{1cm} (NIRC)
\]

\[
1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \quad \hspace{1cm} (\kappa(s))
\]

\[
1 + \tilde{\kappa} \geq \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s
\]

Approach: Split problem

- Solve problem for given \(\tilde{\kappa}\).
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

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V(\tilde{\kappa}) = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
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s.t. \(y_1 - c_1 - \sum q(s)c_2(s) = L^*(\{\kappa(s)\}, \tilde{\kappa})\) (IRC)

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(NIRC)

\[1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s\] (\(\kappa(s)\))

Approach: **Split problem**

- Solve problem for given \(\tilde{\kappa}\). Check ignored constraints
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

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s.t. \(y_1 - c_1 - \sum q(s)c_2(s) = L^*(\{\kappa(s)\}, \tilde{\kappa})\) (IRC)

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i
\]

\[
1 + \kappa(s) = \frac{q(s)u'_1(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s
\] (\(\kappa(s)\))

Approach: Split problem

- Solve problem for given \(\tilde{\kappa}\). Check ignored constraints
- Solve \(V = \max_{\bar{\kappa}} V(\bar{\kappa}), \quad \bar{\kappa} = \arg\max V(\bar{\kappa})\)
CB must open positive “gaps”

For some \( s, \kappa(s) > 0 \)

Under \( \kappa(s) \leq 0 \)

\[
\sum_{s \in S} p(s) = \sum_{s \in S} q(s) \frac{e_1}{e_2(s)(1 + \kappa(s))} \geq \sum_{s \in S} q(s) \frac{e_1}{e_2(s)} = \frac{1 + \Delta(i)}{1 + i}
\]

Since \( \Delta(i) > 0 \),

\[
\left[ \sum_{s \in S} p(s) \right]^{-1} < (1 + i)
\]

Interest rate is too low relative to NIRC.
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In fact, CB always finds optimal to set $\kappa(s) > 0$ for all $s$
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back
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: 1  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)
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  \]

- Replicate that payoff abroad:

  Cost today:  
  Benefit tomorrow:  
  \[
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  - Note $\Delta(i) > 0 \iff \sum_{s \in S} q(s)(e_1(1 + i) \frac{1}{e_2(s)}) > 1$
A key condition: arbitrage return on risk-free bond

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- If \( \sum_s q(s) \left( \left( \frac{e_1}{\sum_s p(s)} \right)^{(1+i)-1} \right) \frac{1}{e_2(s)} \) \( \neq 1 \)

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Characterizing equilibria: Balance of Payment

- Trade deficits and net foreign assets:

\[ c_1 - y_1 = \sum_s \frac{p(s)a^*(s)}{e_1} - \sum_s q(s)[f(s) + F(s)] \]
Take a given \((i, e_1, \{e_2(s)\})\)

**Equilibrium**

HH’s consumption, \((c_1, \{c_2(s)\})\), and asset positions, \((\{a(s), f(s)\})\); Intermediaries consumption, \({d^*_1, d^*_2(s)}\), and asset positions \((\{a^*(s), f^*(s)\})\); central bank transfers \((\{T_2(s)\})\), asset and liabilities \((\{A(s), F(s)\})\); and domestic asset prices \((p(s))\), such that:

1. HH and Intermediaries maximize taking prices as given,
2. the central bank budget constraint holds, and
3. the domestic financial markets clear:

\[
a(s) + a^*(s) + A(s) = 0 \quad \forall s \in S
\]